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横观各向同性饱和地基轴对称 Biot 固结问题的解析解

An analysis of axisymmetric consolidation for transversely isotropic saturated soils

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摘 要: 基于 Biot 理论研究了横观各向同性饱和地基半空间的轴对称固结问题。通过引入状态变量, 构造出一组描述土体固结的等价状态变量微分方程。利用 Hankel- Laplace 联合变换对其进行求解, 获得饱和土骨架位移、应力、孔隙水压力及渗流量的一般积分形式解, 由此分析横观各向同性半空间饱和地基受荷载作用的固结性状。文末给出了数值算例。

关键词: Biot 固结; 横观各向同性饱和地基; Hankel- Laplace 联合变换

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Abstract: Based on Biot's general theory, the axisymmetric consolidation problems of transversely isotropic saturated soils are studied. By employing two state variables, the governing consolidation equations are reduced to a set of equivalent partial differential equations which can be solved by means of Hankel- Laplace integral transform technique. The general solutions for displacements and stress components of the solid matrix, the fluid pressure and discharge are obtained in integral form. At the end of this paper, the consolidation settlement of the transversely isotropic saturated soils is discussed and also a numerical result is presented.

Key words: Biot's general theory of consolidation; transversely isotropic saturated soils; Hankel- Laplace integral transform

1 引 言*

Biot[1] 1941 年从严格的固结机理出发, 推导了正确反映孔隙水压力消散与土骨架变形相互关系的各向同性土体的多维固结方程, 建立了"真三维固结理论"。许多学者利用 Biot 理论研究饱和土的固结问题。Mc Namee & Gibson[2,3] 和 Schiffman 等[4] 通过积分变换并引入势函数分析了二维固结平面应变问题和轴对称问题, 并将其推广至三维非轴对称固结问题, 成功地得到了这一课题的解析解。Cheng 等[5] 则基于边界积分方程方法(BIEM), 给出了这一问题的数值解。

以上文献均假设土为匀质、各向同性的饱和多孔介质。考虑到天然土体由于历史沉积作用而具有明显的横观各向同性特性, 这种假设似欠合理。而对于横观各向同性饱和土体, 虽然 Biot[6] 早在 50 年代就给出了普遍的固结方程组, 但由于这些方程数学结构上的复杂性, 使得横观各向同性饱和土体这一地基模型在实际应用中受到了很大的限制。如何建立横观各向同性饱和土普遍固结方程的一般解进而了解土体的固结性状, 国内外至今尚未见报道。

本文从横观各向同性饱和土体的 Biot 固结方程出发, 引入状态变量, 建立了相应问题的状态变量微分方程组, 然后通过 Hankel- Laplace 联合变换, 得到了饱和土骨架位移、应力、孔隙水压力及渗流量在变换域内的解, 而其物理域中的真实解可由相应的逆变换数值给出。文末给出的数值算例分析了横观各向同性饱和土体在固结过程中的位移沉降特性。

2 横观各向同性饱和土的 Biot 固结方程及其解

考虑横观各向同性饱和地基半空间在竖向圆形荷载作用下的轴对称固结问题。不妨令 $r-\theta$ 平面代表各向同性平面, z 轴沿地基深度方向。则在柱坐标系中, 考虑孔隙水可压缩情形下饱和土体的 Biot 固结方程:

平衡方程
$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{zr}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0 \tag{1a}$$

$$\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}=0 \tag{1b}$$

应力- 应变关系

$$\sigma_r=(2B_1+B_2)\frac{\partial u_r}{\partial r}+B_2\frac{u_r}{r}+B_3\frac{\partial u_z}{\partial z}-B_6\varepsilon \tag{2a}$$

$$\sigma_\theta=B_2\frac{\partial u_r}{\partial r}+(2B_1+B_2)\frac{u_r}{r}+B_3\frac{\partial u_z}{\partial z}-B_6\varepsilon \tag{2b}$$

$$\sigma_z=B_4\frac{\partial u_z}{\partial z}+B_3\frac{\partial u_r}{\partial r}+B_3\frac{u_r}{r}-B_7\varepsilon \tag{2c}$$

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$$\tau_{zr} = B_5\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{2d}$$

$$\sigma_f = B_6 \frac{\partial u_r}{\partial r} + B_6 \frac{u_r}{r} + B_7 \frac{\partial u_z}{\partial z} - B_8 \varepsilon \tag{2e}$$

渗流方程

$$-\kappa_r \frac{\partial \sigma_f}{\partial r} = \frac{\partial w_r}{\partial t} \tag{3a}$$

$$-\kappa_z \frac{\partial \sigma_f}{\partial z} = \frac{\partial w_z}{\partial t} \tag{3b}$$

式中　$\sigma_r, \sigma_\theta, \sigma_z$ 分别为饱和土体的径向、切向及竖向总应力; τ_{zr} 为土体的剪应力; σ_f 为孔隙水压力(以压缩为正); u_r, u_z 分别为土骨架的径向和竖向位移; w_r, w_z 分别为孔隙水的相对径向和竖向位移, 即 $w_r = f(v_r - u_r)$, $w_z = f(v_z - u_z)$, 其中 v_r, v_z 分别为孔隙水的径向和竖向位移, f 为孔隙率; $\varepsilon = \frac{\partial w_r}{\partial r} + \frac{w_r}{r} + \frac{\partial w_z}{\partial z}$; $\kappa_r = \frac{K_r}{\gamma_w}$, $\kappa_z = \frac{K_z}{\gamma_w}$, 其中 K_r, K_z 分别为饱和土的径向、竖向渗透系数, γ_w 为水的容重; $B_j (j = 1, 2, \cdots, 8)$ 是饱和土介质的弹性常数, 它们还可由骨架的六个弹性常数 C_{11}, C_{12}, C_{13}, C_{33}, C_{44}, C_{66} ($C_{11} = C_{12} + 2C_{66}$)、颗粒体积模量 K_s 和孔隙水体积模量 K_f 表示为[7]

$$\left.\begin{aligned}
&B_1 = C_{66}, B_2 = C_{12} + B_6^2/B_8, B_3 = C_{33} + B_6 B_7/B_8,\\
&B_4 = C_{33} + B_7^2/B_8, B_5 = C_{44},\\
&B_6 = -[1 - (C_{11} + C_{12} + C_{13})/(3K_s)]B_8,\\
&B_7 = -[1 - 2(C_{13} + C_{33})/(3K_s)]B_8,\\
&B_8 = \left[\frac{1-f}{K_s} + \frac{f}{K_f} - \frac{2C_{11} + 2C_{12} + 4C_{13} + C_{33}}{9K_s^2}\right]^{-1}
\end{aligned}\right\} \tag{4}$$

为方便分析, 引入无量纲参数及无量纲变量 $\bar{r} = \frac{r}{r_0}$, $\bar{z} = \frac{z}{r_0}$, $\bar{u}_r = \frac{u_r}{r_0}$, $\bar{u}_z = \frac{u_z}{r_0}$, $\bar{w}_r = \frac{w_r}{r_0}$, $\bar{w}_z = \frac{w_z}{r_0}$, $\bar{B}_j = \frac{B_j}{q}$ ($j = 1, 2, \cdots, 8$), $\bar{\sigma}_f = \frac{\sigma_f}{q}$, $\bar{\sigma}_r = \frac{\sigma_r}{q}$, $\bar{\sigma}_\theta = \frac{\sigma_\theta}{q}$, $\bar{\sigma}_z = \frac{\sigma_z}{q}$, $\bar{\tau}_{zr} = \frac{\tau_{zr}}{q}$, $\bar{\kappa}_z = \frac{\kappa_z}{\kappa_r}$, $\bar{t} = \frac{\kappa_r q t}{r_0^2}$, 其中 r_0 为荷载作用半径, q 为荷载的分布强度。则式(1a)~(3b)可化为

$$\frac{\partial \bar{\sigma}_r}{\partial \bar{r}} + \frac{\partial \bar{\tau}_{zr}}{\partial \bar{z}} + \frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{\bar{r}} = 0 \tag{1a$'$}$$

$$\frac{\partial \bar{\sigma}_z}{\partial \bar{z}} + \frac{\partial \bar{\tau}_{zr}}{\partial \bar{r}} + \frac{\bar{\tau}_{zr}}{\bar{r}} = 0 \tag{1b$'$}$$

$$\bar{\sigma}_r = (2\bar{B}_1 + \bar{B}_2)\frac{\partial \bar{u}_r}{\partial \bar{r}} + \bar{B}_2 \frac{\bar{u}_r}{\bar{r}} + \bar{B}_3 \frac{\partial \bar{u}_z}{\partial \bar{z}} - \bar{B}_6 \varepsilon \tag{2a$'$}$$

$$\bar{\sigma}_\theta = \bar{B}_2 \frac{\partial \bar{u}_r}{\partial \bar{r}} + (2\bar{B}_1 + \bar{B}_2)\frac{\bar{u}_r}{\bar{r}} + \bar{B}_3 \frac{\partial \bar{u}_z}{\partial \bar{z}} - \bar{B}_6 \varepsilon \tag{2b$'$}$$

$$\bar{\sigma}_z = \bar{B}_4 \frac{\partial \bar{u}_z}{\partial \bar{z}} + \bar{B}_3 \frac{\partial \bar{u}_r}{\partial \bar{r}} + \bar{B}_3 \frac{\bar{u}_r}{\bar{r}} - \bar{B}_7 \varepsilon \tag{2c$'$}$$

$$\bar{\tau}_{zr} = \bar{B}_5\left(\frac{\partial \bar{u}_r}{\partial \bar{z}} + \frac{\partial \bar{u}_z}{\partial \bar{r}}\right) \tag{2d$'$}$$

$$\bar{\sigma}_f = \bar{B}_6 \frac{\partial \bar{u}_r}{\partial \bar{r}} + \bar{B}_6 \frac{\bar{u}_r}{\bar{r}} + \bar{B}_7 \frac{\partial \bar{u}_z}{\partial \bar{z}} - \bar{B}_8 \varepsilon \tag{2e$'$}$$

$$-\frac{\partial \bar{\sigma}_f}{\partial \bar{r}} = \frac{\partial \bar{w}_r}{\partial \bar{t}} \tag{3a$'$}$$

$$-\bar{\kappa}_z \frac{\partial \bar{\sigma}_f}{\partial \bar{z}} = \frac{\partial \bar{w}_z}{\partial \bar{t}} \tag{3b$'$}$$

将式(2a$'$), (2b$'$), (2d$'$), (2e$'$)代入式(1a$'$), 将式(2c$'$), (2d$'$), (2e$'$)代入式(1b$'$), 得到

$$b_1 \frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{u}_r}{\partial \bar{r}} + \frac{\bar{u}_r}{\bar{r}}\right) + \frac{1}{b_5}\frac{\partial \bar{u}_r^2}{\partial \bar{z}^2} + b_3 \frac{\partial \bar{u}_z^2}{\partial \bar{r}\partial \bar{z}} + \frac{1}{b_5}\frac{\partial \bar{u}_z^2}{\partial \bar{r}\partial \bar{z}} + b_6 \frac{\partial \bar{\sigma}_f}{\partial \bar{r}} = 0 \tag{5a}$$

$$\left(b_3 + \frac{1}{b_5}\right)\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \bar{u}_r}{\partial \bar{r}} + \frac{\bar{u}_r}{\bar{r}}\right) + \frac{1}{b_5}\nabla^2 \bar{u}_z + b_4 \frac{\partial \bar{u}_z^2}{\partial \bar{z}^2} + b_7 \frac{\partial \bar{\sigma}_f}{\partial \bar{z}} = 0 \tag{5b}$$

由式(3a$'$), (3b$'$), (2e)我们有

$$\nabla^2 \bar{\sigma}_f + \bar{\kappa}_z \frac{\partial^2 \bar{\sigma}_f}{\partial \bar{z}^2} = b_8 \frac{\partial \bar{\sigma}_f}{\partial \bar{t}} - b_6 \frac{\partial}{\partial \bar{t}}\left(\frac{\partial \bar{u}_r}{\partial \bar{r}} + \frac{\bar{u}_r}{\bar{r}}\right) - b_7 \frac{\partial \bar{u}_z^2}{\partial \bar{z}\partial \bar{t}} \tag{5c}$$

式(5a)~(5c)是以 $\bar{u}_r, \bar{u}_z, \bar{\sigma}_f$ 为基本未知变量的横观各向同性饱和土体的固结方程, 其中 $b_1 = 2\bar{B}_1 + \bar{B}_2 - \frac{\bar{B}_6^2}{\bar{B}_8}$, $b_3 = \bar{B}_3 - \frac{\bar{B}_6 \bar{B}_7}{\bar{B}_8}$, $b_4 = \bar{B}_4 - \frac{\bar{B}_7^2}{\bar{B}_8}$, $b_5 = \frac{1}{\bar{B}_5}$, $b_6 = \frac{\bar{B}_6}{\bar{B}_8}$, $b_7 = \frac{\bar{B}_7}{\bar{B}_8}$, $b_8 = \frac{1}{\bar{B}_8}$, $\nabla^2 = \frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}$。

为求解上述固结方程, 引入两个中间变量 $\bar{u}_v = \frac{\partial \bar{u}_r}{\partial \bar{r}} + \frac{\bar{u}_r}{\bar{r}}$, $\bar{\tau}_{vz} = \frac{\partial \bar{\tau}_{rz}}{\partial \bar{r}} + \frac{\bar{\tau}_{rz}}{\bar{r}}$。它们和土骨架竖向位移 $\bar{u}_z$、正应力 $\bar{\sigma}_z$、无量纲孔隙水竖向渗流量 $\bar{Q}$ ($\bar{Q} = \frac{Q r_0}{\kappa_r q}$, Q 为竖向渗流量)及孔隙水压力 $\bar{\sigma}_f$ 一起构成了一组状态变量, 以这 6 个变量为基本未知变量, 经整理, 则式(5a)~(5c)可化为如下的偏微分方程组

$$\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\begin{bmatrix} \bar{u}_z \\ \bar{u}_v \\ \bar{Q} \\ \bar{\sigma}_z \\ \bar{\tau}_{vz} \\ \bar{\sigma}_f \end{bmatrix} = \begin{bmatrix} 0 & -\frac{b_3}{b_4} & 0 & \frac{1}{b_4} & 0 & -\frac{b_7}{b_4} \\ -\nabla^2 & 0 & 0 & 0 & b_5 & 0 \\ 0 & M & 0 & \frac{\partial}{\partial \bar{t}}\frac{b_7}{b_4} & 0 & N \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & P & 0 & -\nabla^2 \frac{b_3}{b_4} & 0 & Q \\ 0 & 0 & -\frac{1}{\bar{\kappa}_z} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_z \\ \bar{u}_v \\ \bar{Q} \\ \bar{\sigma}_z \\ \bar{\tau}_{vz} \\ \bar{\sigma}_f \end{bmatrix} \tag{6}$$

式中　$M = \frac{\partial}{\partial \bar{t}}\left(b_6 - \frac{b_3 b_7}{b_4}\right)$; $N = \nabla^2 - \frac{\partial}{\partial \bar{t}} b_8 - \frac{\partial}{\partial \bar{t}}\frac{b_7^2}{b_4}$; $P = \nabla^2\left(-b_1 + \frac{b_3^2}{b_4}\right)$; $Q = -\nabla^2\left(b_6 - \frac{b_3 b_7}{b_4}\right)$。

式(6)和式(5a)~(5c)本质上是相同的。和方程(5a)~(5c)相比,式(6)增加了三个未知函数,但同时该偏微分方程组的阶次却由二阶降为一阶。因此,引入状态变量的实质在于,将固结方程一律归化为一阶微分方程来处理,而宁可使方程和未知变量增加。下面将看到,这种转化对本文固结方程的求解是有益的。

方程(6)可通过积分变换方法来求解。记函数 $f(\bar{r}, \bar{t})$ 的 v 阶 Hankel- Laplace 联合变换为 $\hat{\bar{f}}^v(p, s) = \int_0^\infty\int_0^\infty \bar{r} f(\bar{r}, \bar{t}) J_v(\bar{r}p) e^{-s\bar{t}} d\bar{r} d\bar{t}$,这里符号"~"表示 Hankel 变换,符号"^"表示 Laplace 变换。对方程(6)施以零阶 Hankel- Laplace 联合变换,可将其化为一阶常微分方程组,方程组的矩阵形式为

$$\frac{d\tilde{\Psi}^0}{d\bar{z}} = \boldsymbol{A} \cdot \tilde{\Psi}^0 \tag{7}$$

式中　$\tilde{\Psi}^0(p, \bar{z}, s) = \{\hat{\bar{u}}_z^0(p, \bar{z}, s), \hat{\bar{u}}_v^0(p, \bar{z}, s), \hat{\bar{Q}}^0(p, \bar{z}, s), \hat{\bar{\sigma}}_z^0(p, \bar{z}, s), \hat{\bar{\tau}}_{vz}^0(p, \bar{z}, s), \hat{\bar{\sigma}}_f^0(p, \bar{z}, s)\}^{\mathrm{T}}$, $\boldsymbol{A} =$

$$\begin{vmatrix} 0 & -\frac{b_3}{b_4} & 0 & \frac{1}{b_4} & 0 & -\frac{b_7}{b_4} \\ p^2 & 0 & 0 & 0 & b_5 & 0 \\ 0 & s\left(b_6 - \frac{b_3 b_7}{b_4}\right) & 0 & s\frac{b_7}{b_4} & 0 & -p^2 - b_8 s - \frac{b_7^2}{b_4}s \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & p^2\left(b_1 - \frac{b_3^2}{b_4}\right) & 0 & p^2\frac{b_3}{b_4} & 0 & p^2\left(b_6 - \frac{b_3 b_7}{b_4}\right) \\ 0 & 0 & -\frac{1}{\bar{\kappa}_z} & 0 & 0 & 0 \end{vmatrix}$$。

矩阵 $\boldsymbol{A}$ 的特征方程为

$$\lambda^6 + \gamma_2 \lambda^4 + \gamma_3 \lambda^2 + \gamma_4 = 0 \tag{8}$$

式中系数:

$$\gamma_2 = \frac{-b_4 p^2 + 2b_3 \bar{\kappa}_z p^2 + b_3^2 b_5 \bar{\kappa}_z p^2 - b_1 b_4 b_5 \bar{\kappa}_z p^2 - b_7^2 s - b_4 b_8 s}{b_4 \bar{\kappa}_z},$$

$\gamma_3 = \{-2b_3 p^4 - b_3^2 b_5 p^4 + b_1 b_4 b_5 p^4 + b_1 \bar{\kappa}_z p^4 + b_4 b_5 b_6^2 p^2 s - 2b_6 b_7 p^2 s - 2b_3 b_5 b_6 b_7 p^2 s + b_1 b_5 b_7^2 p^2 s - 2b_3 b_8 p^2 s - b_3^2 b_5 b_8 p^2 s + b_1 b_4 b_5 b_8 p^2 s\}/(b_4 \bar{\kappa}_z)$,

$$\gamma_4 = \frac{-b_1 p^6 - b_6^2 p^4 s - b_1 b_8 p^4 s}{b_4 \bar{\kappa}_z}。$$

由于 Laplace 逆变换的围道积分将在 s 复平面内进行,故应将式(8)视为复系数一元六次方程来处理(实质上是复系数三次方程,但解的表达式和实系数三次方程有所不同)。其 6 个根 $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3$ ($\mathrm{Re}[\lambda_j] \geqslant 0$) 可表达为

$$\lambda_1^2 = \left(-\frac{y}{2} + \sqrt{\frac{y^2}{4} + \frac{x^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{x}{3}\right)\Big/\left(-\frac{y}{2} + \sqrt{\frac{y^2}{4} + \frac{x^3}{27}}\right)^{\frac{1}{3}} - \frac{\gamma_2}{3} \tag{9a}$$

$$\lambda_2^2 = \left(-\frac{y}{2} + \sqrt{\frac{y^2}{4} + \frac{x^3}{27}}\right)^{\frac{1}{3}} \cdot \omega + \left(-\frac{x}{3}\right)\Big/\left\{\omega \cdot \left(-\frac{y}{2} + \sqrt{\frac{y^2}{4} + \frac{x^3}{27}}\right)^{\frac{1}{3}}\right\} - \frac{\gamma_2}{3} \tag{9b}$$

$$\lambda_3^2 = \left(-\frac{y}{2} + \sqrt{\frac{y^2}{4} + \frac{x^3}{27}}\right)^{\frac{1}{3}} \cdot \omega^2 + \left(-\frac{x}{3}\right)\Big/\left\{\omega^2 \cdot \left(-\frac{y}{2} + \sqrt{\frac{y^2}{4} + \frac{x^3}{27}}\right)^{\frac{1}{3}}\right\} - \frac{\gamma_2}{3} \tag{9c}$$

其中 $x = -\frac{1}{3}\gamma_2^2 + \gamma_3$, $y = \frac{2\gamma_2^3}{27} - \frac{1}{3}\gamma_2\gamma_3 + \gamma_4$, $\omega = \frac{-1 + \sqrt{3}i}{2}$;根式 $\sqrt{\frac{y^2}{4} + \frac{x^3}{27}}$ 代表实部为正的一个单值分支,而 $\left(-\frac{y}{2} + \sqrt{\frac{y^2}{4} + \frac{x^3}{27}}\right)^{\frac{1}{3}}$ 可取三次根式的任一单值分支。

根据常微分方程理论[8],并注意到 $\lim_{\bar{z}\to\infty}\tilde{\Psi}^0(p, \bar{z}, s) = 0$,我们可写出式(7)的解

$$\tilde{\Psi}^0(p, \bar{z}, s) = \sum_{j=1}^{3} c_j \boldsymbol{h}_j e^{-\lambda_j \bar{z}} \tag{10}$$

式中　c_j ($j = 1, 2, 3$) 为 p, s 的任意函数, $\boldsymbol{h}_j = \{h_{j1}, h_{j2}, h_{j3}, h_{j4}, h_{j5}, h_{j6}\}^{\mathrm{T}}$ ($j = 1, 2, 3$) 为矩阵 $\boldsymbol{A}$ 对应于特征根 $-\lambda_j$ 的特征向量,即满足

$$\begin{vmatrix} \lambda_j & -\frac{b_3}{b_4} & 0 & \frac{1}{b_4} & 0 & -\frac{b_7}{b_4} \\ p^2 & \lambda_j & 0 & 0 & b_5 & 0 \\ 0 & s\frac{\delta_1}{b_4} & \lambda_j & s\frac{b_7}{b_4} & 0 & \delta_3 \\ 0 & 0 & 0 & \lambda_j & -1 & 0 \\ 0 & p^2\frac{\delta_2}{b_4} & 0 & p^2\frac{b_3}{b_4} & \lambda_j & p^2\frac{\delta_1}{b_4} \\ 0 & 0 & -\frac{1}{\bar{\kappa}_z} & 0 & 0 & \lambda_j \end{vmatrix} \cdot \begin{vmatrix} h_{j1} \\ h_{j2} \\ h_{j3} \\ h_{j4} \\ h_{j5} \\ h_{j6} \end{vmatrix} = 0 \tag{11}$$

式中, $\delta_1 = b_4 b_6 - b_3 b_7$, $\delta_2 = b_1 b_4 - b_3^2$, $\delta_3 = -p^2 - b_8 s - b_7^2/b_4 s$。显然,方程(11)的解为

$$\left|\begin{array}{l} h_{j1} = -\frac{\alpha_j}{\Delta_j} h_{j6}, \; h_{j2} = \frac{\beta_j}{\Delta_j} h_{j6}, \; h_{j3} = \lambda_j \bar{\kappa}_z h_{j6} \\ h_{j4} = \frac{p^2 \alpha_j - \lambda_j \beta_j}{b_5 \lambda_j \Delta_j} h_{j6} \\ h_{j5} = \frac{p^2 \alpha_j - \lambda_j \beta_j}{b_5 \Delta_j} h_{j6} \end{array}\right. \tag{12}$$

其中

$\alpha_j = \lambda_j(b_3 b_7 p^2 + \delta_1 p^2 + b_3 b_5 \delta_1 p^2 - b_5 b_7 \delta_2 p^2 + b_4 b_7 \lambda_j^2)$,

$\beta_j = p^2(b_3 b_7 p^2 + \delta_1 p^2 + b_4 b_7 \lambda_j^2 - b_4 b_5 \delta_1 \lambda_j^2)$,

$\Delta_j = -b_3^2 p^4 - \delta_2 p^4 - 2b_3 b_4 p^2 \lambda_j^2 + b_4 b_5 \delta_2 p^2 \lambda_j^2 - b_4^2 \lambda_j^4$。

将式(12)代入式(10), 可进一步写出

$$\tilde{\Psi}^0(p, \bar{z}, s) = \sum_{j=1}^{3} \tilde{c}_j \tilde{\boldsymbol{h}}_j e^{-\lambda_j \bar{z}} \tag{13}$$

这里 $\tilde{\boldsymbol{h}}_j = \{-\alpha_j, \beta_j, \lambda_j \bar{\kappa}_z \Delta_j, \frac{p^2\alpha_j - \lambda_j\beta_j}{b_5 \lambda_j}, \frac{p^2\alpha_j - \lambda_j\beta_j}{b_5}, \Delta_j\}^{T}$, $\tilde{c}_j$ 仍为p, s 的任意函数。

至此, 我们已经获得了饱和土骨架位移、应力、土体渗透流量及孔隙水压力在 Hankel- Laplace 变换域内的形式解, 为确定表达式中的三个待定函数 $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, 必须考虑边界条件。

3　边值条件

前面已经指出, 本文探讨横观各向同性饱和地基半空间受竖向圆形荷载作用的固结问题。假定地基表面具有完全的透水能力, 且受突然施加的分布压力 $qH(t)H(r_0 - r)$ 作用, 其中 H 为 Heaviside 单位阶梯函数。则饱和地基固结的无量纲边界条件可由下面的式子表示

$$\bar{\sigma}_z(\bar{r}, 0, \bar{t}) = -H(\bar{t})H(1-\bar{r}) \quad 0 \leqslant \bar{r} \leqslant \infty, -\infty \leqslant \bar{t} \leqslant +\infty \tag{14a}$$

$$\bar{\tau}_{rz}(\bar{r}, 0, \bar{t}) = \bar{\tau}_{zr}(\bar{r}, 0, \bar{t}) = 0 \quad 0 \leqslant \bar{r} \leqslant \infty, -\infty \leqslant \bar{t} \leqslant +\infty \tag{14b}$$

$$\bar{\sigma}_f(\bar{r}, 0, \bar{t}) = 0 \quad 0 \leqslant \bar{r} \leqslant \infty, -\infty \leqslant \bar{t} \leqslant +\infty \tag{14c}$$

对式(14)进行零阶 Hankel- Laplace 变换, 并结合式(13), 我们可写出

$$\hat{\bar{\sigma}}_z^0(p, 0, s) = \frac{p^2\alpha_1 - \lambda_1\beta_1}{b_5\lambda_1}\tilde{c}_1 + \frac{p^2\alpha_2 - \lambda_2\beta_2}{b_5\lambda_2}\tilde{c}_2 + \frac{p^2\alpha_3 - \lambda_3\beta_3}{b_5\lambda_3}\tilde{c}_3 = -\frac{J_1(p)}{sp} \tag{15a}$$

$$\hat{\bar{\tau}}_{rz}^0(p, 0, s) = \frac{p^2\alpha_1 - \lambda_1\beta_1}{b_5}\tilde{c}_1 + \frac{p^2\alpha_2 - \lambda_2\beta_2}{b_5}\tilde{c}_2 + \frac{p^2\alpha_3 - \lambda_3\beta_3}{b_5}\tilde{c}_3 = 0 \tag{15b}$$

$$\hat{\bar{\sigma}}_f^0(p, 0, s) = \Delta_1\tilde{c}_1 + \Delta_2\tilde{c}_2 + \Delta_3\tilde{c}_3 = 0 \tag{15c}$$

故

$$\tilde{c}_1 = -\frac{1}{\delta b_5}\{(p^2\alpha_2 - \lambda_2\beta_2)\Delta_3 - (p^2\alpha_3 - \lambda_3\beta_3)\Delta_2\}\frac{J_1(p)}{ps} \tag{16a}$$

$$\tilde{c}_2 = -\frac{1}{\delta b_5}\{(p^2\alpha_3 - \lambda_3\beta_3)\Delta_1 - (p^2\alpha_1 - \lambda_1\beta_1)\Delta_3\}\frac{J_1(p)}{ps} \tag{16b}$$

$$\tilde{c}_3 = -\frac{1}{\delta b_5}\{(p^2\alpha_1 - \lambda_1\beta_1)\Delta_2 - (p^2\alpha_2 - \lambda_2\beta_2)\Delta_1\}\frac{J_1(p)}{ps} \tag{16c}$$

其中 $\delta = \frac{p^2\alpha_1 - \lambda_1\beta_1}{b_5^2\lambda_1}\{(p^2\alpha_2 - \lambda_2\beta_2)\Delta_3 - (p^2\alpha_3 - \lambda_3\beta_3)\Delta_2\} + \frac{p^2\alpha_2 - \lambda_2\beta_2}{b_5^2\lambda_2}\{(p^2\alpha_3 - \lambda_3\beta_3)\Delta_1 - (p^2\alpha_1 - \lambda_1\beta_1)\Delta_3\} + \frac{p^2\alpha_3 - \lambda_3\beta_3}{b_5^2\lambda_3}\{(p^2\alpha_1 - \lambda_1\beta_1)\Delta_2 - (p^2\alpha_2 - \lambda_2\beta_2)\Delta_1\}$。

将式(16a)~(16c)反代入式(13), 进行零阶 Hankel- Laplace 反演, 即可获得各状态变量在真实物理域内的解。这里仅给出工程中比较关心的地基表面的竖向位移积分表达式:

$$\bar{u}_z(\bar{r}, 0, \bar{t}) = \int_{Br}\int_0^{\infty} \frac{1}{b_5\delta}\{\alpha_1(p^2\alpha_2 - \lambda_2\beta_2)\Delta_3 - \alpha_1(p^2\alpha_3 - \lambda_3\beta_3)\Delta_2 + \alpha_2(p^2\alpha_3 - \lambda_3\beta_3)\Delta_1 - \alpha_2(p^2\alpha_1 - \lambda_1\beta_1)\Delta_3 + \alpha_3(p^2\alpha_1 - \lambda_1\beta_1)\Delta_2 - \alpha_3(p^2\alpha_2 - \lambda_2\beta_2)\Delta_1\}\frac{J_1(p)}{s}J_0(p\bar{r})e^{s\bar{t}}\,dp\,ds \tag{17}$$

式中“$\int_{Br}$”是 Bromwich 围道积分。

4　数值算例分析

由式(17)即可求得横观各向同性饱和土体在固结过程中的位移沉降。由于该积分表达式是含有相当复杂被积函数的双重广义积分, 故只能通过数值积分来完成。对此笔者利用“Mathematica”数学软件编制了相应的程序, 其中涉及的 Laplace 反演采用 Schapery[9] 提出的方法, 即$\bar{u}_z^0(p, 0, \bar{t}) = s\hat{\bar{u}}_z^0(p, 0, s)\big|_{s=\frac{1}{2\bar{t}}}$。

在本节给出的数值算例中, 取横观各向同性饱和土的物理力学参数为 $C_{11} = 10^7$ Pa, $C_{12} = 2\times10^6$ Pa, $C_{13} = 2.4\times10^6$ Pa, $C_{33} = 0.8\times10^7$ Pa, $C_{44} = 3.2\times10^6$ Pa, $C_{66} = 4\times10^6$ Pa, $K_s = 3\times10^7$ Pa, $K_f = 2\times10^6$ Pa, $f = 0.2$, $\kappa_r = 10^{-7}(\mathrm{m^4 \cdot s^{-1} \cdot N^{-1}})$, $\kappa_z = 10^{-8}(\mathrm{m^4 \cdot s^{-1} \cdot N^{-1}})$, $q = 10^7$ Pa。相应的无量纲参数为 $\bar{B}_1 = 0.4$, $\bar{B}_2 = 0.7806$, $\bar{B}_3 = 1.3314$, $\bar{B}_4 = 1.2866$, $\bar{B}_5 = 0.32$, $\bar{B}_6 = -0.6912$, $\bar{B}_7 = -0.6327$, $\bar{B}_8 = 0.8228$, $\bar{\kappa}_z = 0.1$。则由式(17)可计算地基表面的固结沉降。这里定义固结度为

$$U(\bar{r}, 0, \bar{t}) = \frac{\bar{u}_z - \bar{u}_z|_{\bar{t}=0}}{\bar{u}_z|_{\bar{t}=\infty} - \bar{u}_z|_{\bar{t}=0}} \tag{18}$$

图 1 给出了圆形荷载作用下饱和地基表面中心点的固结度随无量纲时间 $\bar{t}$ 的变化曲线。

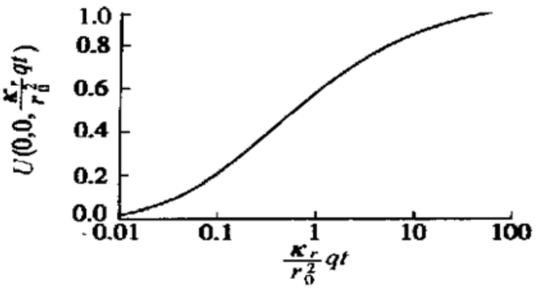

图 1 固结度随时间的变化曲线

Fig. 1 Degree of consolidation $U(0, 0, \bar{t})$ varying with dimensionless time $\bar{t}$

5 结 语

本文从横观各向同性饱和土体的 Biot 固结方程出发, 通过引入状态变量, 建立了一组等价的饱和地基固结状态变量微分方程组。该微分方程可采用 Hankel-Laplace 积分变换技术来求解, 从而获得饱和土骨架位移、应力、孔隙水压力及渗透流量的一般积分形式解。文末结合具体边值问题给出了数值分析实例。本文推导过程简洁、严密, 思路明晰, 首次给出了横观各向同性饱和地基半空间在圆形荷载作用下 Biot 固结问题的解析解, 为今后类似课题的研究奠定了基础。

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苏通长江公路大桥开始初步设计

我国长江下游最后一座大桥 ——苏通长江公路大桥即将进行初步设计。

苏通长江公路大桥位于江苏境内长江南通段, 上游距 1999 年 9 月建成的江阴长江公路大桥约 82 km, 下游距长江入海口约 108 km。其北岸为南通市, 南岸为苏州所辖常熟市, 作为江苏省东部靠近上海的过江通道, 对缩短苏北沿海地区到上海的行车时间, 加快苏南、苏北地区经济发展, 减轻过江交通压力, 完善江苏省高速公路网具有重要意义。

苏通大桥江面宽, 河势复杂, 基岩埋藏深, 通航标准高。在国家交通部和中央江苏省委、省政府的直接领导下, 经过前段较长时间的审慎细致的预可行性及可行性研究, 对桥址、桥型等作了多方案比选和技术经济分析, 2001 年 6 月国家发展计划委员会下达了经国务院批准的项目建议书, 苏通大桥正式立项。

拟建的苏通大桥全长约 7.6 km, 主航道桥为双塔双索面斜拉桥, 主跨 1088 m, 为世界最长的斜拉桥, 索塔高度近 300 m。公路等级按平原微丘区全封闭双向六车道高速公路设计, 计算行车速度 100 km/h, 大桥标准宽度 34.0 m, 通航净空单孔双航道 891 m × 62 m, 主墩基础初拟采用大型钢沉井基础。

最近, 苏通大桥初步设计招标评标工作在江苏省南通市进行。鉴于该大桥技术复杂, 难度大, 涉及的专业面广, 且招标文件要求投标人须有国外设计/咨询机构合作, 因此评标委员会成员系由业主直接从与投标人无利益关系的国内技术经济专家中选定, 并邀请了熟悉国外咨询机构的瑞典公路局桥梁专家参加。现评标工作已结束, 按照国家交通部有关规定, 将由业主定标, 然后即开始进行初步设计。

苏通大桥预计将于 2003 年第一季度开工, 2008 年底建成。

(杭州地下工程研究中心办公室 供稿)