Free vibrations of tapered piles embedded partially in elastic foundations
部分埋入弹性地基的变截面桩自由振动

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Abstract The numerical methods for calculating the natural frequencies and the mode shapes of the tapered piles embedded partially in the elastic foundations are developed. The ordinary differential equation governing the free vibrations of such piles are derived, in which the effect of axial load is considered. The Runge- Kutta method and the determinant search method combined with the Regula- Falsi method are used for integrating the differential equation and determining the eigenvalues(natural frequencies), respectively.
Key words determinant search method, elastic foundation, free vibration, mode shape, pile embedded partially, Regula- Falsi method, Runge - Kutta method, tapered pile
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文 摘 阐述了部分埋入弹性地基的变截面桩的自振频率和振型的数值计算方法,并推导了桩顶作用轴向荷载的变截面桩的控制自由振动的微分方程,最后把此微分方程应用于 Runge- Kutta 法和 Regula- Falsi 法,采用特征值搜索法算出桩的自振频率和振型。
关键词 特征值搜索法,弹性地基,自由振动,振型,部分埋入桩,Regula- Falsi 法,Runge- Kutta 法,变截面桩
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1 Introduction*

Since piles are used often to support the superstructures which are frequently subjected to the dynamic loadings, it is very important to analysis the free vibrations of piles embedded partially or fully in the elastic foundations. The problem of free vibrations of beams/piles resting on the elastic foundations has been investigated by many researchers[1~4].

In the analysis of foundation problems, it is very difficult to model the foundation behavior mathematically due to the complexity of foundation properties. In the past decades several models of elastic foundations had been developed such as Winkler model, Pasternak model, Filonenko-Borodich model and etc. In this study, the Winkler model, under which the elastic foundation is depicted as a large number of closely spaced translational springs, is adopted as the foundation model.

The main purpose of this paper is to investigate the free vibrations of piles embedded partially in the elastic foundations. The differential equation governing the free vibration of such piles is derived, in which the effects of not only taper of piles but also axial load on natural frequencies are included. This equation subjected to its boundary conditions is solved numerically for calculating the natural frequencies and the corresponding mode shapes. The Runge- Kutta method is used for integrating the differential equation, and the determinant search method combined with the Regula- Falsi method is used for obtaining the natural frequencies. In the numerical examples, the two lowest frequencies are calculated for clamped-clamped, clamped-hinged and clamped-free end constraints with the various system parameters: embedded length, foundation modulus, section ratio and axial load. Also typical mode shapes of vibrating piles are presented.

2 Mathematical model

The symbols and load of a tapered pile are defined in Figure 1(a), which is embedded partially in an elastic foundation. The embedded end is clamped and the other is clamped or hinged or free. Its span length and embedded length are l and αL, respectively. The axial load is depicted as P, in which the compressive load is positive. The width, area and area moment of inertia of cross-section of tapered pile at any distance x with its origin at the embedded end are depicted as d, A and I, respectively. The values of d, A and I at the end a($x = 0$) are depicted as d_a, A_a and I_a, respectively. Also the value I at the end b($x = l$) is expressed as I_b. The foundation modulus which is defined as the load per unit area required to produce unit displacement of the foundation is depicted as K. Figure 1 (b) shows the typical mode shape of a vibrating pile, in which w is the relative amplitude in free vibration. In this figure, it is noted that the term of $q(= Kdw)$ at distance x is the restoring force per unit pile length due to the elastic foundation.

The partial differential equation governing the free vibration of tapered beam element resting on an elastic foundation with an axial force is given by Timoshenko et al.[5]

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and from this equation, the ordinary differential equation is obtained as follows.

$$EIw'''' + 2EI'w''' + (EI'' + P)w'' - \rho\omega^2 Aw + Kdw = 0 \qquad (1)$$

where $(') = \mathrm{d}/\mathrm{d}x$, E is Young's modulus, ρ is mass density of beam material and ω is angular frequency.

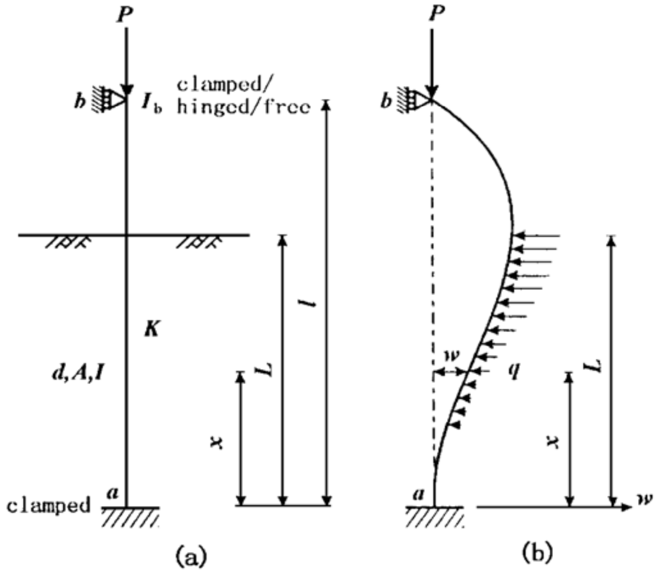

Fig. 1 (a) The symbols of pile embedded partially in an elastic foundation with shear layer and (b) its typical mode shape

Now consider the variable width d, variable area A and variable area moment of inertia of cross-section I of tapered pile at any distance x. Many taper types for the piles are available in the foundation engineering. In this study, the linear tapers whose first order dimension of cross-section varies with linear functional fashion are chosen as the taper types of piles. The d, A and I of such tapers at distance x given by Kupta[6] are as follows.

$$d = d_a[1 + (\lambda^e - 1)x/l] \qquad (2)$$

$$A = A_a[1 + (\lambda^{1/n} - 1)x/l]^m \qquad (3)$$

$$I = I_a[1 + (\lambda^{1/n} - 1)x/l]^n \qquad (4)$$

in which λ is section ratio defined by a ratio of I_b to I_a, that is

$$\lambda = I_b/I_a \qquad (5)$$

Also in equations (2) ~ (4), the constant values e, m and n can be determined when the cross-sectional shapes of linearly tapered piles are given. In this study, the linear taper with the solid circular cross-section is chosen, which is frequently called as the cone type pile. For this taper, the corresponding e, m and n values are obtained by simple mathematical calculations as follows.

$$e = 1/4 \qquad (6a)$$

$$m = 2 \qquad (6b)$$

$$n = 4 \qquad (6c)$$

It is recalled that the values of e, m and n for another taper can be obtained easily if the cross sectional shapes for linear taper are given[7].

The following equations are obtained by differentiating equation (4) once and twice.

$$I' = n(\lambda^{1/n} - 1)l^{-1}I_a[1 + (\lambda^{1/n} - 1)x/l]^{n-1} \qquad (7)$$

$$I'' = n(n-1)(\lambda^{1/n} - 1)^2 l^{-2} I_a[1 + (\lambda^{1/n} - 1)x/l]^{n-2} \qquad (8)$$

To facilitate the numerical studies and to obtain the most general results for this class of problem, the following non-dimensional system variables are defined.

$$\xi = x/l \qquad (9)$$

$$n = w/l \qquad (10)$$

$$\alpha = \alpha L/l \qquad (11)$$

$$\mu = Kd_a l^4/(\pi^4 EI_a) \qquad (12)$$

$$p = Pl^2/(\pi^2 EI_a) \qquad (13)$$

When equations (2) ~ (4), (7) and (8) are substituted into equation (1) and the non-dimensional forms of equations (9) ~ (13) are used, the result is

$$\eta^{\mathrm{iv}} = -\frac{2n(\lambda^{1/n} - 1)}{1 + (\lambda^{1/n} - 1)\xi}\eta^{\mathrm{iii}} - \left|\frac{n(n-1)(\lambda^{1/n} - 1)^2}{[1 + (\lambda^{1/n} - 1)\xi]^2} + \frac{\pi^2 p}{[1 + (\lambda^{1/n} - 1)\xi]^n}\eta^{\mathrm{ii}}\right| + \left|c_i^2[1 + (\lambda^{1/n} - 1)\xi]^{(m-n)} - \frac{\pi^4 \mu[1 + (\lambda^e - 1)\xi]}{[1 + (\lambda^{1/n} - 1)\xi]^n}\right|\eta, \quad \mu = 0 \, for \, \xi > \alpha \qquad (14)$$

where $(^{\mathrm{i}}) = \mathrm{d}/\mathrm{d}\xi$ and the ith frequency parameter c_i is defined as follows. It is noted that since the pile is not embedded for $\xi > \alpha$ the corresponding μ is zero in equation (14).

$$c_i = \omega_i l^2(\rho A_a/EI_a)^{1/2}, \quad i = 1, 2, 3, \cdots \qquad (15)$$

Now the boundary conditions are considered. For the clamped end placed at $x = 0$ or $x = l$, both the amplitude $w_{a,b}$ and the rotation $\theta_{a,b}(= w'_{a,b})$ are zero and those non-dimensional forms are as follows.

$$\eta = 0 \text{ at } \xi = 0 \text{ or } \xi = 1 \qquad (16a)$$

$$\eta^{\mathrm{i}} = 0 \text{ at } \xi = 0 \text{ or } \xi = 1 \qquad (16b)$$

For the hinged end placed at $x = l$, both the amplitude w_b and the bending moment $M_b(= -EIw''_b)$ are zero and those non-dimensional forms are as follows.

$$\eta = 0 \text{ at } \xi = 1 \qquad (17a)$$

$$\eta^{\mathrm{ii}} = 0 \text{ at } \xi = 1 \qquad (17b)$$

Finally for the free end placed at $x = l$, the bending moment M_b is zero and the shear force $V_b(= M'_b)$ is Pw'_b, and those non-dimensional forms are as follows.

$$\eta^{\mathrm{ii}} = 0 \text{ at } \xi = 1 \qquad (18a)$$

$$\eta^{\mathrm{iii}} + \pi^2 p \lambda^1 \eta^{\mathrm{i}} = 0 \text{ at } \xi = 1 \qquad (18b)$$

In the aforementioned theoretical analysis, the ordinary differential equation governing the free vibration of linearly tapered piles embedded partially in the elastic foundation and subjected to the axial load, and the boundary conditions are derived as the non-dimensional forms.

3 Numerical method and validation

Based on the above analysis, a general FORTRAN 77 computer program is written to calculate the frequency parameters c_i and the corresponding mode shapes (ξ, η) for given geometries of piles i. e. embedded length parameter α, foundation parameter μ, section ratio λ, end constraint and load parameter p. The Runge-Kutta method is used for integrating the differential equation, and the determinant search method combined with the Regula-Falsi method is used for obtaining the eigenvalues(frequency parameters c_i). The algorithm developed herein to solve the differential equation has two convergence criteria. For the sake of completeness, this algorithm is summarized as follows.

(1) Specify geometry of pile: α, μ, λ, end constraint and p.

(2) According to the specified end constraint, set two homogeneous boundary conditions at $\xi = 0$ which are equations (16a) and (16b).

(3) Consider fourth order system, equation (14), as two initial value problems whose initial values are two homogeneous boundary conditions at $\xi = 0$, as chosen in above step. And assume a trial frequency parameter c_i in which the first trial value is zero.

(4) Using the Runge-Kutta method, integrate equation (14) from $\xi = 0$ to $\xi = 1$. Perform two separate integrations, one for each of two chosen boundary conditions.

(5) From the Runge-Kutta solutions, evaluate the determinant D of the coefficient matrix at $\xi = 1$ for the chosen set of two homogeneous boundary conditions. If $D = 0$, then the trial value of c_i is an eigenvalue. The first convergence criterion is

$$|D| \leqslant tol_1 \tag{19}$$

where tol_1 is a tolerance. When the first convergence criterion is met, terminate the calculations, and print c_i and(ξ, η).

(6) If equation (19) is not satisfied, then increase c_i and repeat steps (4) and (5). Note the sign of D in each iteration. If D changes the sign between two consecutive trials of c_a and c_b, then the eigenvalue lies between these last two trials.

(7) Using the Regula-Falsi method, compute the advanced trial of c_c based on two previous values of c_a and c_b as follows, and repeat calculations.

$$c_c = \frac{c_a|D_b| + c_b|D_a|}{|D_a| + |D_b|} \tag{20}$$

where D_a and D_b are the corresponding values of D at c_a and c_b.

(8) Second convergence criterion is

$$\left|\frac{c_a - c_d}{c_c}\right| \leqslant tol_2 \tag{21}$$

where tol_2 is a tolerance and c_d is the c_i value which satisfies $D_c \times D_d < 0$. When this criterion is met, terminate the calculations and print c_i and (ξ, η).

For the studies, suitable convergence of solutions are obtained for a step size $\Delta\xi = 1/200$ with $tol_1 = 1 \times 10^{-12}$ and $tol_2 = 1 \times 10^{-6}$ in the Runge-Kutta method.

In the numerical examples of this study, two lowest frequency parameters c_i for the three end constraints of clamped-clamped(cc), clamped-hinged(ch) and clamped-free(cf) with the solid circular cross-section($e = 1/4$, $m = 2$ and $n = 4$) are calculated for given geometries of piles.

For comparison purpose, finite element solutions based on the commercial package SAP 90 are used to compute c_i for case with $\alpha = 0.7$, $\mu = 50.0$, $\lambda = 1.5$, and $p = 0$. All the results computed by this study and SAP 90 are compared in Table 1. These results show that the frequency parameters c_i of this study quite agree with those of SAP 90, and such comparisons serve to validate both the theory and numerical method developed herein.

Table 1　Comparisons of c_i① between this study and SAP 90

end contraint	c_1			c_2		
	this study	SAP 90	error② /%	this study	SAP 90	error② /%
c_c	67.35	67.24	0.16	83.44	83.43	0.01
c_h	52.79	52.69	0.19	79.78	79.78	0.00
c_f	15.08	15.06	0.13	66.20	66.20	0.30

① $\alpha = 0.7$, $\mu = 50.0$, $\lambda = 1.5$, $p = 0$, $e = 1/4$, $m = 2$, $n = 4$;
② error(%) = $(|c_{i,(\text{this study})} - c_{i,(\text{SAP 90})}| / c_{i,(\text{SAP 90})}) \times 100$

4 Numerical results and discussion

The results shown in Figures 2~6, all for the solid circular cross-sections with three end constraints, depict the variations of c_i ($i = 1, 2$) with α, μ, λ and p, respectively. In Figure 2, $\mu = 50.0$, $\lambda = 1.5$, and $p = 0$ and the c_i always increases as the embedded length parameter α increases.

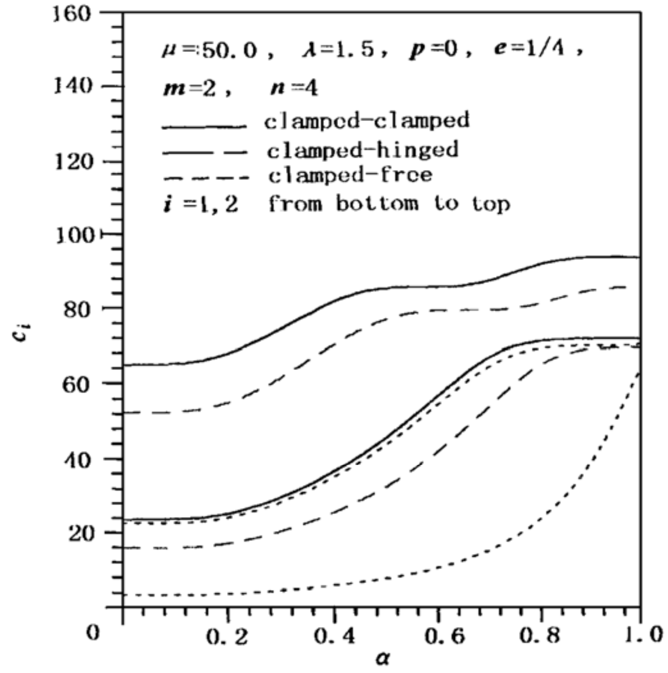

Fig. 2　c_i **versus α curves**

Figure 3 shows the relationships between c_i and μ for $\alpha = 0.7$, $\lambda = 1.5$ and $p = 0$. The trend is as expected all c_i increase as the foundation parameter μ increases. Particularly the increasing rate of c_1 for clamped-free end constraint is very small when μ is greater than about 20 and therefore the frequency parameter c_1 approaches a horizontal asymptote.

Shown in Figure 4 are the c_i versus λ curves with $\alpha = 0.7$, $\mu = 50.0$ and $p = 0$. It is noted that several c_i curves reach highest or lowest points as the section ratio λ increases. All increasing rates of c_i are very small when λ is greater than about 0.5. In case of clamped-free end constraint, it is noted that the effects of λ on c_i is negligible practically.

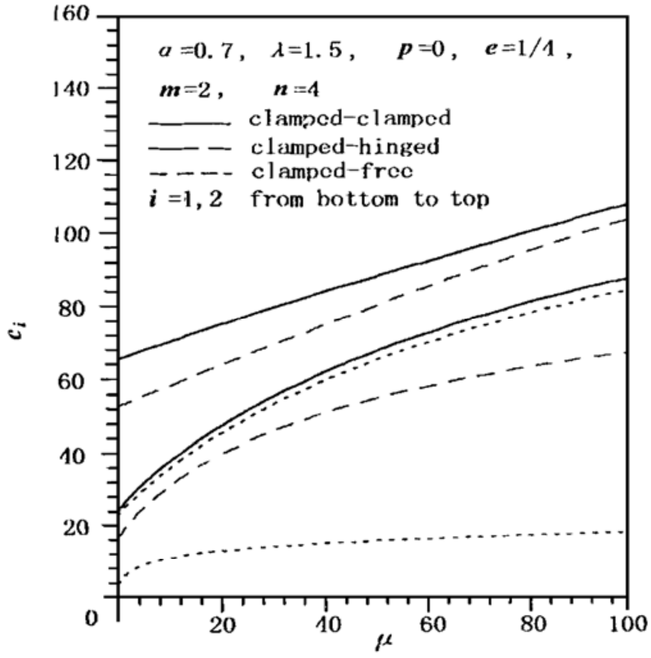

Fig. 3 c_i **versus** μ **curves**

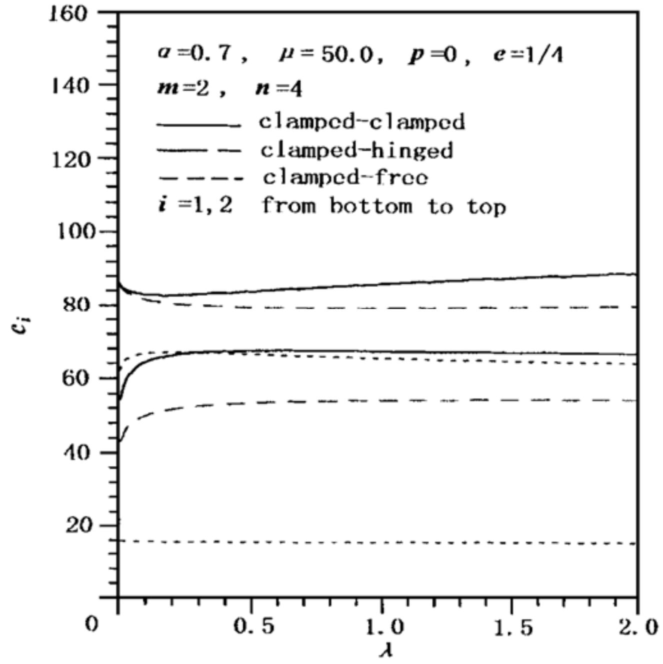

Fig. 4 c_i **versus** λ **curves**

Figure 5 depicts the relationships between c_i and p for $\alpha = 0.7$, $\mu = 50.0$ and $\lambda = 1.5$. In this figure, it is recalled that the compressive load is positive. The trend is as expected: all c_i decrease as the load parameter p increases. When the c_i vanishes, the pile buckles at the corresponding p value marked by ■ on the p axis. After buckling of piles, the second frequency parameters c_2 are meaningless, which are not presented in this figure. It is concluded that from this figure, the p values marked by ■ are the buckling load parameters of the piles whose geometries are given as the legends at the top of this figure.

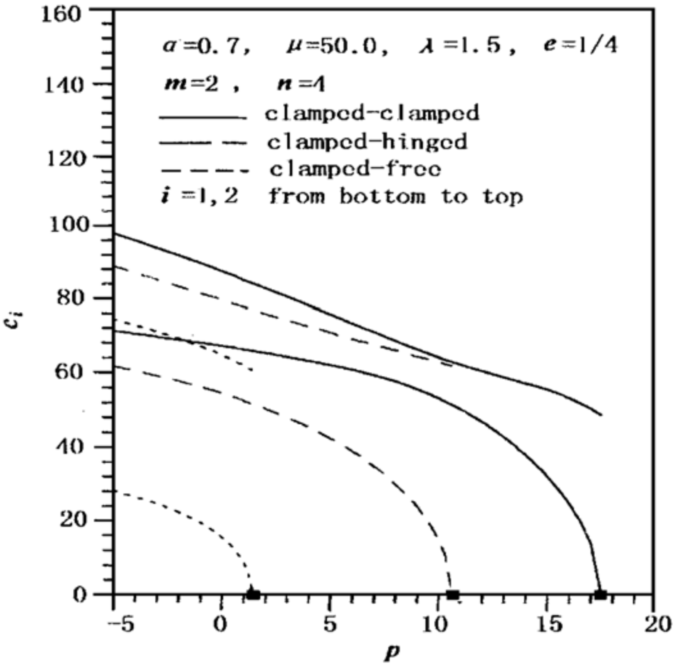

Fig. 5 c_i **versus** p **curves**

The typical mode shapes of the vibrating piles computed herein are shown in Figure 6, based on $\alpha = 0.7$, $\mu = 50.0$, $\lambda = 1.5$ and $p = 1$. In these mode shapes, the nodal points are marked by ■, whose relative transverse amplitudes are zero. To increase the natural frequencies, the axis of pile should be restrained usually by the struts. For example of clamped-clamped end constraint, if a strut is placed at $\xi = 0.55$ marked by ■ which is a nodal point of the second mode, the value of c_1 is increased from 66.4 to 85.4 since the first mode is impossible due to its transverse restraint.

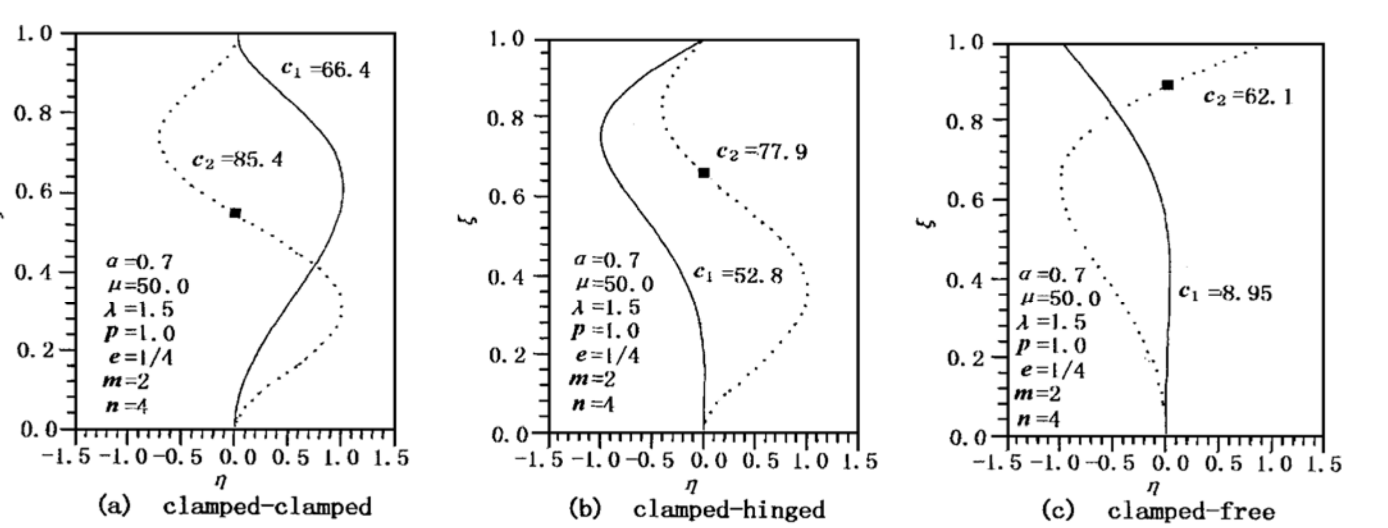

Fig. 6 Example of mode shapes

5 Concluding remarks

The numerical methods are developed for calculating the natural frequencies and the mode shapes of tapered piles which are partially embedded in the elastic foundations. The ordinary differential equation governing the free vibrations of such piles is derived, in which the effects of taper types and the axial forces on the natural frequencies are included. This equation subjected to the boundary conditions is solved numerically. The numerical results obtained by this study and the commercial package SAP 90 are agreed quite well with each other.

The present numerical methods are found to be efficient, accurate, and highly versatile in the practical ranges of system parameters: embedded length parameter α, foundation parameter μ, section ratio λ, and load parameter p. Since the frequency parameters with the corresponding mode shapes, in which each of these system parameters can be considered in separate or in combination, may be calculated, it is expected that the numerical methods developed herein should be used practically in the foundation engineering fields.

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